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RESONANT PHENOMENA ACCOMPANYING THE SCATTERING OF A PLANE WAVE ON AN INHOMOGENEOUS LAYER AND ON A PACK OF INHOMOGENEOUS LAYERS

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The problem of scattering of a plane monochromatic wave impinging obliquely on an inhomogeneous plane parallel layer or on a pack of identical inhomogeneous layers, is considered. The inhomogeneity of the medium is described by the dependence of its refractive index n on the coordinate x orthogonal to the layer boundaries. The angle of incidence of the wave is such, that total internal reflection occurs. The case in which the corresponding differential equation has two inflection points, is studied. The coefficients of transmission and reflection are computed for the high frequency region, and it is shown that resonant transmissions are possible when the function $n(x)$ exhibits certain symmetry properties, i.e the whole layer is "transparent" for certain angles of incidence even though these angles exceed the critical angle.

1. Formulation of the problem of scattering on a layer. We distinguish in the three-dimensional space three regions: the layer $-1 < x < 1$, $-\infty < y, z < \infty$, and two half-spaces, $x < -1$ and $x > 1$. We assume that both half-spaces are filled with a homogeneous isotropic medium with the refractive index $n_0 > 0$, and that the layer is filled with an isotropic, but inhomogeneous medium with the refractive index $n(x) > 0$ depending only on the coordinate x varying across the layer. The field $U(x, y, z, t)$ satisfies the wave equation

$$n^2(x)\frac{\partial^2 U}{\partial t^2} = \Delta U \tag{1.1}$$

and $n(x) = n_0$ when $|x| < 1$. The problem is assumed plane, i.e. $U \equiv U(x, y, t)$. We formulate at the boundaries $x = \pm 1$ the conditions of continuity of the field and of its normal derivative. We assume that in the half-space $x < -1$ the field $U(x, y, t)$ is the superposition of a plane monochromatic wave impinging on the layer at the angle α to the normal, and of the wave reflected from the layer, while in the half-space $x > 1$ we have the transmitted wave only, i.e.

$$U(x, y, t) = \exp[-i\omega(t - n_0 y \sin\alpha - n_0(x+1)\cos\alpha)] + R\exp[-i\omega(t - n_0 y\sin\alpha + n_0(x+1)\cos\alpha)], \quad x < -1 \tag{1.2}$$

$$U(x, y, t) = C\exp[-i\omega(t - n_0 y\sin\alpha - n_0(x-1)\cos\alpha], \quad x > 1$$

The constants R and C denote the reflection and transmission coefficients. In the problem of scattering on a layer these coefficients must be determined, and the function $U(x, y, t)$ must satisfy the equation (1.1) and the specified conditions of continuity. Below we make certain assumptions concerning the function $n(x)$ for obtaining asymptotic formulas for the reflection and transmission coefficients with $\omega \to \infty$.

Putting

$$U(x, y, t) = \exp[-i\omega(t - n_0 y \sin\alpha)]\, V(x) \tag{1.3}$$

we obtain the following equation for the function $V(x)$ for $|x| < 1$:

$$V''(x) = \omega^2 q(x)\, V(x), \; q(x) = n_0^2 \sin^2\alpha - n^2(x) \tag{1.4}$$

When $x < -1$ and $x > 1$, this function is defined by the expressions (1.2) and (1.3) and must, in addition, be continuously differentiable for all x.

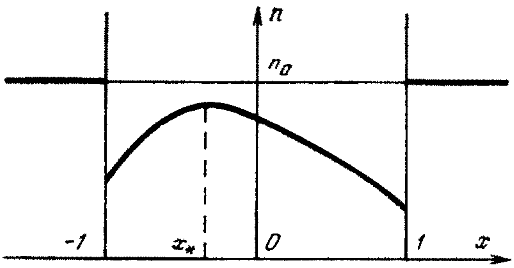

Fig. 1

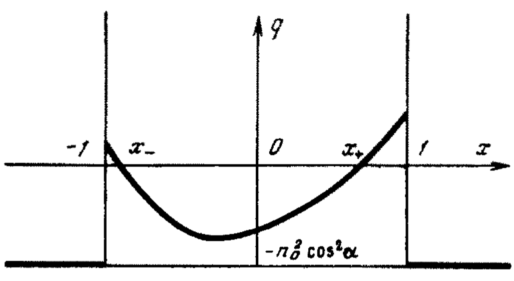

Fig. 2

For constructing the asymptotic formulas for R and C , we require asymptotic formulas for the solutions of (1.4); the form of these formulas depends substantially on the properties of the function $q(x)$.

We assume that the refractive index $n(x)$ has a maximum within the layer at some point $x = x_*$, $|x_*| < 1$, and is a monotone function at all remaining $x \in [-1, 1]$ (more exactly, we require that $n'(x) \neq 0$ when $x \neq x_*$, $x \in [-1, 1]$) and is also sufficiently smooth (see Fig. 1). The requirement that $n'''(x)$ must be continuous for $|x| < 1$ is sufficient for obtaining the principal terms of the asymptotic formulas, while the stipulation for $n^{(5)}(x)$ to be continuous is sufficient for the first order corrections.

The above assumption about the function $n(x)$ implies the following properties of the function $q(x)$. For small α (e.g., at the normal wave incidence) we have $q(x) < 0$ when $|x| < 1$. If however $n_0 > n_{-1} \equiv n(-1 + 0)$ then for the angles α such that $n_0 \sin\alpha > n_1$ we can find an interval near the point $x = -1$ on which $q(x) > 0$. Similarly, if $n_0 > n_1 \equiv n(1 - 0)$ and $n_0 \sin\alpha > n_{-1}$, then such an interval will appear near the point $x = 1$.

We shall assume that the angle of incidence of the wave is such, that

$$\max(n_{-1}, n_1) < n_0 \sin\alpha < n(x_*) \tag{1.5}$$

Function $q(x)$ is depicted in Fig. 2. The condition (1.5) ensures that, when $|x| < 1$, the function $q(x)$ has two simple zeros x and x_+. In other words, the equation (1.4) has two simple points of inflection x_- and x_+.

Let us now turn our attention to the terminology. In the intervals in which $q(x) < 0$, the equation (1.4) has oscillating solutions, therefore we shall call these intervals the transmittance intervals. The intervals in which $q(x) > 0$ shall be called the opacity intervals or barriers. The interval (x_-, x_+) lying between the barriers shall be called a well.

At small α the layer is transparent to the wave. In this case the coefficients R and C are of the order of unity, a part of the energy of the incident wave passes into the half-space $x > 1$ in the form of the transmitted wave, and a part is reflected from the layer. If, on the other hand, $\alpha > \alpha_*$ where α_* is the critical angle defined by the relation $n_0 \sin \alpha_* = \min (n_{-1}, n_1)$, the opaque intervals (barriers) appear in the layer and the coefficient C is generally exponentially small (the effect of total internal reflection [1]; when the angles are supercritical, there is practically no penetration of energy into the half-space $x > 1$. We shall show in Sect. 4 that under certain conditions imposed on $n(x)$ and α the wave can be transmitted also at angles greater than the critical angle. Apparently this phenomenon is connected with the resonance of the wave in the well between the barriers.

2. Transition matrix. Let us introduce the vectors $\mathbf{z} = (V(x), \omega^{-1} V'(x))$, $\mathbf{r} = (R + 1, i\sigma (R - 1))$, $\mathbf{c} = (C, - i\sigma C)$, where $\sigma = n_0 \cos \alpha$ the vectors are column vectors, (but from now on shall be written in a horizontal line in order to conserve space). Equation (1.4) yields the following system of differential equations for the vector $\mathbf{z}(x)$:

$$\mathbf{z}'(x) = \omega \begin{Vmatrix} 0 & 1 \\ q(x) & 0 \end{Vmatrix} \mathbf{z}(x), \quad |x| < 1 \tag{2.1}$$

and (1.2), (1.3) and the continuity of $V(x)$ and $V'(x)$ yield the boundary conditions

$$\mathbf{z}(-1) = \mathbf{r}, \ \mathbf{z}(1) = \mathbf{c} \tag{2.2}$$

Let $Z(x)$ be the fundamental matrix of the system (2.1). In this case we have, for $|x| < 1$, $\mathbf{z}(x) = Z(x) \boldsymbol{\beta}$ where $\boldsymbol{\beta}$ is a constant vector. Using (2.2) to eliminate $\boldsymbol{\beta}$, we obtain the following (inhomogeneous) system of two linear equations:

$$\mathbf{r} = T\mathbf{c} \quad (T = Z(-1) Z^{-1}(1)) \tag{2.3}$$

for the exact values of the coefficients of reflection R and transmission C.

It is clear that the matrix T (which shall be called the transition matrix) is independent of the choice of the fundamental matrix $Z(x)$. Consequently we can choose, as $Z(x)$, a matrix with known asymptotic properties at $\omega \to \infty$. The fundamental matrix of the system (2.1) can be constructed by various methods on the interval containing two simple points of inflection. In the present case we shall use the method of matching uniform asymptotic formulas containing the Airy functions. We fix $x_0 \in (x_-, x_+)$ arbitrarily, and obtain two intervals $I_- = [-1, x_0)$ and $I_+ = [x_0, 1]$, each of which contains exactly one point of inflection. Let $\eta^{(l)}(x) = (\eta_1^{(l)}(x), \eta_2^{(l)}(x))$, $l = 1, 2$ be two linearly independent vector solutions of (2.1) on I_-, and $\zeta^{(l)}(x) = (\zeta_1^{(l)}(x), \zeta_2^{(l)}(x))$, $l = 1, 2$ two linearly independent vector solutions of (2.1) on I_+. We shall use them to construct the fundamental matrices $Z_-(x) = (\eta^{(1)}(x), \eta^{(2)}(x))$, $Z_+(x) = (\zeta^{(1)}(x), \zeta^{(2)}(x))$. and take, as $Z(x)$, the matrix

$$Z(x) = \begin{cases} Z_-(x) Z_-^{-1}(x_0), & -1 < x \leqslant x_0 \\ Z_+(x) Z_+^{-1}(x_0), & x_0 \leqslant x < 1 \end{cases}$$

continuous at $|x| < 1$.

The transition matrix T can be written in terms of the functions $\eta_k^{(l)}(x)$, $\zeta_k^{(l)}(x)$, l, $k = 1, 2$, and the calculations show that

$$T = \frac{1}{\Delta_-(x_0)\Delta_+(1)} \sum_{i,k=1}^{2} w_{ik} A_{ik}, \quad \Delta_\pm(x) = \det Z_\pm(x) \tag{2.4}$$

$$w_{ik} = (-1)^{i+k}[\eta_2^{(3-i)}(x_0)\zeta_1^{(3-k)}(x_0) - \eta_1^{(3-i)}(x_0)\zeta_2^{(3-k)}(x_0)]$$

$$A_{ik} = \begin{Vmatrix} -\eta_1^{(i)}(-1)\zeta_2^{(k)}(1) & \eta_1^{(i)}(-1)\zeta_1^{(k)}(1) \\ -\eta_2^{(i)}(-1)\zeta_2^{(k)}(1) & \eta_2^{(i)}(-1)\zeta_1^{(k)}(1) \end{Vmatrix}, \quad \det A_{ik} = 0$$

Note that the functions $\eta_1^{(l)}(x)$, $\zeta_1^{(l)}(x)$ $(l = 1, 2)$ are solutions of the equation (1.4), and that relations are valid for w_{ik} : $w_{ik} = (-1)^{i+k}\omega^{-1} W[\zeta_1^{(3-k)}, \eta_1^{(3-i)}]$, where $W[f, g]$ denote the Wronskian in f and g. This implies that w_{ik} is independent of x_0. Since $\Delta_-(x_0)$ is also independent of x_0 (being the determinant of the fundamental matrix), it follows that so is the transition matrix T.

3. Asymptotic formulas. The formula (2.4) yields an expression for the transition matrix in terms of four solutions of the system (2.1). The solutions should be chosen in such a manner, that the asymptotic formulas [2] uniform on $I_\pm$ hold as $\omega \to \infty$. These formulas can be expressed in vector form by

$$\eta^{(l)}(x) = \begin{Vmatrix} \mu_l^-(x)(1 + o_2) + o_2\mu_l^{-\prime}(x) \\ \omega^{-1}\mu_l^{-\prime}(x)(1 + o_2) + o_1\mu_l^-(x) \end{Vmatrix}, \tag{3.1}$$

$$\zeta^{(l)}(x) = \begin{Vmatrix} \mu_l^+(x)(1 + o_3) + o_2\mu_l^{-\prime}(x) \\ \omega^{-1}\mu_l^{+\prime}(x)(1 + o_2) + o_1\mu_l^+(x) \end{Vmatrix}$$

$$\mu_1^\pm(x) = \frac{\omega^{1/6}}{\sqrt{|\varphi_\pm'|}} u(\omega^{2/3}\varphi_\pm(x)), \quad \mu_2^\pm(x) = \frac{\omega^{1/6}}{\sqrt{|\varphi_\pm'|}} v(\omega^{2/3}\varphi_\pm(x))$$

$$\varphi_\pm(x) = \left(\frac{3}{2}\int_{x_\pm}^{x} |q(\tau)|^{1/2}\, d\tau\right)^{2/3} \operatorname{sign} q(x)$$

where $u(\tau)$ and $v(\tau)$ are real Airy functions and the symbol o_k replaces, from now on, the symbol $O(\omega^{-k})$.

Substituting (3.1) into (2.4) and using the known asymptotic formulas for the Airy functions [3], we obtain the following asymptotics for the matrix T

$$T = \frac{1}{\sqrt{\varkappa_1\varkappa_{-1}}}\Big\{w_{11}\exp[(\gamma_+ + \gamma_-)\omega]K_{-+}^{+-} - \tag{3.2}$$

$$-\frac{w_{12}}{2}\exp[(-\gamma_+ + \gamma_-)\omega]K_{--}^{++} - \frac{w_{21}}{2}\exp[(\gamma_+ - \gamma_-)\omega]K_{-+}^{-+} -$$

$$-\frac{w_{22}}{4}\exp[(-\gamma_+ - \gamma_-)\omega]K_{++}^{++}\Big\}, \quad \gamma_\pm = \gamma_\pm(\alpha) = \pm\int_{x_\pm}^{\pm 1}\sqrt{q(\tau)}\, d\tau > 0$$

$$w_{11} = \sin\psi + o_1, \quad w_{12} = -\cos\psi + o_1, \quad w_{21} = -\cos\psi + o_1$$

$$w_{22} = -\sin\psi + o_1$$

$$\psi = \omega J + \pi/2, \quad J = J(\alpha) = \int_{x_-}^{x_+}\sqrt{-q(\tau)}\, d\tau$$

$$K_{-+}^{+-} = \begin{Vmatrix} a_+ & b_- \\ c_- & d_+ \end{Vmatrix}, \quad \begin{matrix} a_\pm = \pm\varkappa_1(1 + o_1), \quad b_\pm = \pm 1 + o_1 \\ c_\pm = \pm\varkappa_1\varkappa_{-1}(1 + o_1), \quad d_\pm = \pm\varkappa_{-1}(1 + o_1) \end{matrix}$$

$$\varkappa_{\pm 1} = \sqrt{q(\pm 1 \mp 0)}$$

It is clear that w_{11} may vanish, and this implies that the first term in (3.2) is not always the principal term in the asymptotics of T. We shall therefore consider two cases.

1°. Let $|w_{11}| \geqslant \delta > 0$. Then the first term in (3.2) is the principal, and (3.2), (2.3) yield

$$C = \left[\frac{2i\sigma\sqrt{\varkappa_1\varkappa_{-1}}}{\sin\psi\,(\varkappa_1 + i\sigma)(\varkappa_{-1} + i\sigma)} + o_1\right] \exp\left[-(\gamma_+ + \gamma_-)\,\omega\right] \tag{3.3}$$

$$R = R_0 + o_1, \quad R_0 = \frac{i\sigma - \varkappa_{-1}}{i\sigma + \varkappa_{-1}}, \quad |R_0| = 1$$

This is the case of the usual total internal reflection, and the amplitude of the wave transmitted by the layer is exponentially small. The presence of two barriers within the layer simply reduces to consecutive attenuation of the transmitted wave at these barriers.

2°. Let $w_{11} = 0$ (the resonance case). In this case we have

$$\psi = \pi m + o_1, \quad w_{12} = (-1)^{m+1} + o_1, \quad w_{21} = (-1)^{m+1} + o_1$$
(m is an integer)

The second or third term in (3.2) is the principal one, depending on the relation between γ_+ and γ_-. Let us assume that $\gamma_+ \neq \gamma_-$. We then have

$$C = (-1)^m \left[\frac{4i\sigma\sqrt{\varkappa_1\varkappa_{-1}}}{(\varkappa_> + i\sigma)(\varkappa_< - i\sigma)} + o_1\right] \exp\left(-|\gamma_+ - \gamma_-|\,\omega\right) \tag{3.4}$$

$$R = \begin{cases} R_0 + o_1, & \gamma_- > \gamma_+ \\ R_0^{-1} + o_1, & \gamma_- < \gamma_+ \end{cases}$$

$$\varkappa_> = \begin{cases} \varkappa_1, & \gamma_+ > \gamma_- \\ \varkappa_{-1}, & \gamma_+ < \gamma_- \end{cases}, \quad \varkappa_< = \begin{cases} \varkappa_{-1}, & \gamma_+ > \gamma_- \\ \varkappa_1, & \gamma_+ < \gamma_- \end{cases}$$

This case differs from the previous one by the exponential index in the formula for C. In the physical terms, we can say that in the case of resonance the presence of two barriers within the layer is important, and the actions of these barriers on the transmitted wave oppose each other.

From (3.4) it is clear that the case in which $w_{11} = 0$ and $\gamma_+ = \gamma_-$, is of particular interest. This is the case when the angle of incidence and the refractive index $n(x)$ are such that the barriers have the same "integral" width. We shall call this case the case of integrally symmetric resonance, and will consider it in Sect. 4.

We conclude Sect. 3 by considering the problem of resonance range. The condition $w_{11} = 0$ yields a discrete set of resonance frequencies.

$$\omega_m = \omega_m(\alpha) = \pi\,(m - {}^1/_2)/J(\alpha) + O(m^{-1}), \quad m \to \infty$$

If instead of using the exact equality $w_{11} = 0$ we assume that

$$w_{11} \exp\left[(\gamma_+ + \gamma_-)\,\omega\right] = O\left(\omega^{-1} \exp\left(-|\gamma_+ - \gamma_-|\,\omega\right)\right)$$

then the first term of (3.2) will also cease to be the principal. From this follows that the formula (3.4) holds not only for the frequencies ω_m, but also in their small neighborhoods described by the relation

$$|\omega - \omega_m| \leqslant O\left(\omega_m^{-1} \exp\left[-2\gamma\omega_m\right]\right), \quad \gamma = \max(\gamma_+, \gamma_-) \tag{3.5}$$

In addition to the resonance range, the first correction term for w_{11} is also of interest. After computing this term, we find that the condition of resonance $w_{11} = 0$ has the form

$$\sin\psi + \omega^{-1} K \cos\psi + o_2 = 0, \quad K = \frac{1}{48} \lim_{b \to x_+ - 0} \lim_{a \to x_- + 0} Q \tag{3.6}$$

$$Q = \int_a^b \frac{q''(\tau)\, d\tau}{(-q(\tau))^{5/2}} - \frac{2q''(x_+)}{(q'(x_+))^{3/2} \sqrt{x_+ - b}} - \frac{2q''(x_-)}{(-q'(x_-))^{3/2} \sqrt{a - x_-}}$$

While calculating this, we have used, in (3.1), the first correction terms given in [2], and the terms of order of ω^{-1} from the asymptotic formulas for the Airy functions [4].

The formula (3.6) yields the first correction term for ω_m

$$\omega_m = \pi (m - 1/2)/J(\alpha) - K (\pi m)^{-1} + O(m^{-2}), \quad m \to \infty \tag{3.7}$$

4. Resonance in the integrally symmetric layer and the symmetric layer. Let us first consider the case of the integrally symmetric resonance, i.e. of $w_{11} = 0$ and $\gamma_+ = \gamma_-$. The asymptotics of the transition matrix has the form

$$T = \frac{(-1)^m}{\sqrt{\varkappa_1 \varkappa_{-1}}} \begin{Vmatrix} o_1 & 1 + o_1 \\ -\varkappa_1 \varkappa_{-1}(1 + o_1) & o_1 \end{Vmatrix} \tag{4.1}$$

Substituting the expression (4.1) into the system (2.3) and solving the latter, we obtain the following expressions for R and C :

$$R = \frac{\sigma^2 - \varkappa_1 \varkappa_{-1}}{\sigma^2 + \varkappa_1 \varkappa_{-1}} + o_1, \quad C = (-1)^m \frac{2i\sigma \sqrt{\varkappa_1 \varkappa_{-1}}}{\sigma^2 + \varkappa_1 \varkappa_{-1}} + o_1 \tag{4.2}$$

The case of $R = 0$, i.e. of the complete absence of a reflected wave, is of interest. Let us consider the problem of zeros of the principal term of R. Simple computation shows that the equation $\sigma^2 - \varkappa_1 \varkappa_{-1} = 0$ has in the interval $[0, \pi/2]$ the following unique root:

$$\alpha^\circ = \arcsin \frac{1}{n_0} \sqrt{\frac{n_0^4 - n_1^2 n_{-1}^2}{2n_0^2 - n_1^2 - n_{-1}^2}} \tag{4.3}$$

and α° satisfies the inequalities (1.5).

Thus, if at $\alpha = \alpha^\circ$ the layer is integrally symmetric, i.e. $\gamma_+(\alpha^\circ) = \gamma_-(\alpha^\circ)$, the layer will be almost nonreflective at all resonant frequencies $\omega = \omega_m(\alpha^\circ)$, $R = o_1$, $|C| = 1 + o_1$. The wave passes through such a layer practically without attenuation regardless of the fact that $\alpha > \alpha_*$, i.e. that the angle of incidence exceeds the critical angle. Generally speaking, it is not clear whether we can assert that an angle, at which $R = 0$, will be found. Indeed, the principal term of R is real, but generally $\operatorname{Im} R \neq 0$. The problem of zeros of R can be solved completely when the layer is prefectly symmetric.

In the case when the layer is perfectly symmetric, i.e. $n(-x) = n(x)$, we have always $\gamma_+ = \gamma_-$, and this is independent of α. This means that all resonances are integrally symmetric and the coefficient C is not accompanied by an exponentially small multiplier when $\omega = \omega_m$.

By virtue of the symmetry of the layer we can take $\zeta^{(l)}(x) = (\eta_1^{(l)}(-x), -\eta_2^{(l)}(-x))$ as the solutions of $\zeta^{(l)}(x)$. Choosing $\zeta^{(l)}(x)$ in this manner, we obtain the following relations for the diagonal elements of the matrices A_{ik} :

$$-\eta_1^{(i)}(-1)\zeta_2^{(k)}(1) = \eta_1^{(i)}(-1)\eta_2^{(k)}(-1), \quad \eta_2^{(i)}(-1)\zeta_1^{(k)}(1) = \eta_2^{(i)}(-1)\eta_1^{(k)}(1)$$

Let us put $x_0 = 0$. Then we have

$$w_{12} = w_{21} = -(\eta_1^{(1)}(0)\,\eta_2^{(2)}(0) + \eta_1^{(2)}(0)\,\eta_2^{(1)}(0))$$

This, together with (2.4), implies that the diagonal elements of the transition matrix T coincide with each other.

Let further $w_{11} = 0$. We write the matrix T in the form

$$T = \frac{(-1)^m}{\sqrt{\varkappa_1\varkappa_{-1}}}\begin{Vmatrix} \varepsilon_{11} & 1+\varepsilon_{12} \\ -\varkappa_1\varkappa_{-1}(1+\varepsilon_{21}) & \varepsilon_{22} \end{Vmatrix} \tag{4.4}$$

where ε_{ik} denote the qunatities of order ω^{-1} and, as we said before, $\varepsilon_{11} = \varepsilon_{22}$. Substituting (4.4) into (2.3) we obtain, in place of (4.2), a more detailed formula fo$R$

$$R = \frac{\sigma^2(1+\varepsilon_{12}) - \varkappa_1\varkappa_{-1}(1+\varepsilon_{21}) + i\sigma(\varepsilon_{11}-\varepsilon_{22})}{\sigma^2(1+\varepsilon_{12}) + \varkappa_1\varkappa_{-1}(1+\varepsilon_{21}) + i\sigma(\varepsilon_{12}+\varepsilon_{22})} \tag{4.5}$$

Remembering that $\varepsilon_{11} = \varepsilon_{22}$ we find that $R = 0$ if

$$\sigma^2 - \varkappa_1\varkappa_{-1} + \sigma^2\varepsilon_{12} - \varkappa_1\varkappa_{-1}\varepsilon_{21} = 0 \tag{4.6}$$

Clearly, $\sigma^2\varepsilon_{12} - \varkappa_1\varkappa_{-1}\varepsilon_{21}$ is a real function of order ω^{-1}; however, computing the first correction terms we find that in fact $\sigma^2\varepsilon_{12} - \varkappa_1\varkappa_{-1}\varepsilon_{21} = O(\omega^{-2})$.

Thus there will be no reflected wave at all if the conditions (3.6) and (4.6) hold. Let us rewrite these conditions, replacing in (4.6) the quantities σ and $\varkappa_{\pm 1}$ by their expressions in terms of α and showing the dependence of the correction terms in ω and α

$$\begin{aligned} &\omega J(\alpha) = \pi(m - 1/2) + o_1(\alpha, \omega) \\ &n_0^2\cos^2\alpha - [(n_0^2\sin^2\alpha - n_1^2)(n_0^2\sin^2\alpha - n_{-1}^2)]^{1/2} + o_2(\alpha, \omega) = 0 \end{aligned} \tag{4.7}$$

The system obtained from (4.7) by discarding the functions $o_1(\alpha, \omega)$ and $o_2(\alpha, \omega)$ has, for every m, the solution $\alpha = \alpha^\circ$, $\omega = \pi(m - 1/2)/J(\alpha^\circ)$. The functions $o_1(\alpha, \omega)$ and $o_2(\alpha, \omega)$ are small when $\omega \to \infty$, therefore the system (4.7) has, for every sufficiently large m, (at least) one solution $\alpha = \alpha_m$, $\omega = \omega_m(\alpha_m)$, and

$$\alpha_m = \alpha^\circ + O(m^{-2});\ \omega_m(\alpha_m) = \pi(m - 1/2)/J(\alpha^\circ) + O(m^{-1}),\quad m \to \infty \tag{4.8}$$

We present the results obtained in the form of a theorem.

Theorem. Let the refractive index $n(x)$ satisfy the conditions formulated in Sect. 1, and let the angle of incidence of the wave α satisfy the inequalities (1.5) and $\omega \gg 1$. Then the following assertions hold for the coefficients of reflection R and transmission C in the problem of scattering (1.1), (1.2).

1°. For every α there exists a discrete sequence of resonant frequencies $\{\omega_m(\alpha)\}$ defined by the formula (3.7) and such that: outside the resonance regions, i.e. when

$|w_{11}| \geqslant \delta > 0$, the asymptotic formulas (3.3) hold: in the neighborhood of (3.5) the asymptotic formulas (3.4) hold when $\gamma_-(\alpha) \neq \gamma_+(\alpha)$, and (4.2) hold when $\gamma_-(\alpha) = \gamma_+(\alpha)$

2°. If at $\alpha = \alpha^\circ$ the layer is integrally symmetric, i.e. if $\gamma_-(\alpha^\circ) = \gamma_+(\alpha^\circ)$, then in the neighborhood (3.5) of the resonant frequencies $\omega_m(\alpha^\circ)$ we have $R = O(m^{-1})$, $|C| = 1 + O(m^{-1})$.

3°. For a perfectly symmetric layer there exists a sequence of angles $\{\alpha_m\}$ such that no reflection takes place at the corresponding resonance frequencies $\omega_m(\alpha_m)$, i.e. $R = 0$ and in addition $\alpha_m = \alpha^\circ + O(m^{-2})$, $\omega_m(\alpha_m) = \omega_m(\alpha^\circ) + O(m^{-1})$.

5. Pack of identical layers. We shall consider a generalization of the problem (1.1), (1.2), assuming that we have a pack of N identical layers and solving the problem of scattering on such pack.

Let the s-th layer s $(s = 1, \ldots, N)$ have the boundaries $x = s - 1$ and $x = s$. Since the layers are identical, $n(x) = n(x + s - 1)$, $s = 1, \ldots, N$. We assume that the function $n(x)$ satisfies the condition of Sect. 1 for $0 < x < 1$. As before, we have the equation (1.1) for the field $U(x, y, t)$ and again formulate the conditions of continuity of the field and of its normal derivative at the boundary $x = s$, $s = 0, 1, \ldots, N$. Outside the pack we formulate for the field $U(x, y, t)$ the assumption analogous to those made in Sect. 1: for $x < 0$, we have the first expression of (1.2), and for $x > N$ the second expression. Finally, let the angle α satisfy, as before, the condition (1.5). We introduce, as in the case of a single layer, the functions $q(x)$ and $V(x)$, and the vector $\mathbf{z}(x)$. We denote by $Z_s(x)$ the fundamental matrix of the system (2.1) for $s - 1 < x < s$, i.e. in the s-th layer. The conditions of continuity of $\mathbf{z}$ at the interface boundaries yield the following system of equations ($\boldsymbol{\beta}_s$ are constant vectors):

$$Z_s(s)\boldsymbol{\beta}_s = Z_{s+1}(s)\boldsymbol{\beta}_{s+1}, \quad s = 1, \ldots, N-1 \qquad (5.1)$$
$$\mathbf{r} = Z_1(0)\boldsymbol{\beta}_1, \quad \mathbf{c} = Z_N(N)\boldsymbol{\beta}_N$$

Eliminating successively from (5.1), all $\boldsymbol{\beta}_s$, we obtain the following system of equations for R and S :

$$\mathbf{r} = T_1 T_2 \ldots T_N \mathbf{c} \quad (T_s = Z_s(s-1) Z_s^{-1}(s)) \qquad (5.2)$$

where T_s is the transition matrix for the s-th layer. Let us now use the assumption that all layers are identical and choose, as the matrix $Z_s(x)$, the matrix $Z_1(x + s - 1)$. In this case we find that $T_s = T_1 \equiv T$ and the system (5.2) assumes the form

$$\mathbf{r} = T^N \mathbf{c} \qquad (5.3)$$

Using the system (5.3) we can obtain formulas analogous to (3.3) and (3.4), which are fairly bulky. It will be more interesting to consider the case of a resonance with an integral or perfect symmetry.

Let us assume that integrally symmetric resonance occurs in every layer. Then the formulas (4.1) in which $\varkappa_{-1}$ has been replaced by $\varkappa_0 \equiv \sqrt{q(+0)}$, holds for the transition matrix T since $x = 0$ is the left boundary of the first layer. It is clear that

$$T^{2p} = (-1)^p (E + O_1), \quad T^{2p+1} = (-1)^p (T + O_1), \quad p = 0, 1, \ldots$$

where E denotes a unit matrix and O_1 is a matrix with elements of order o_1.

Consider the case in which the number of layers is odd, $N = 2p + 1$. Solving the system (5.3) we find, that in this case the second part of the assertion 1° (for $\gamma_- = \gamma_+$) and the assertion 2° of the theorem in Sect. 4 remain valid, with the obvious adjustment of the notation, namely, replacing $\varkappa_{-1}$ by $\varkappa_0$, $n(-1)$ by $n(+0)$ and multiplying the formula for C from (4.2) by $(-1)^p$. If every layer is perfectly symmetric, we can prove by induction that the diagonal elements of the matrix T^{2p+1} are identical, and this implies the validity of the assertion 3°. Thus we find that the theorem proved in Sect. 4 can be extended to the case of a pack containing an odd number of identical layers.

Let us now put $N = 2p$. Then the system (5.3) becomes

$$R + 1 = (1 + o_1) C_1 \qquad (5.4)$$

$$i\sigma (R - 1) = (-i\sigma + o_1) C_1, \quad C_1 = (-1)^p C$$

Solving the system (5.4), we obtain $R = o_1$, $C = (-1)^p + o_1$. Thus in the case of integrally symmetric resonance we have $R = o_1$ for the even number of layers irrespective of whether the relation $\alpha = \alpha^\circ$ holds. The pack composed of an even number of layers is, in a certain sense, more transparent than a single layer or an odd number of layers. This phenomenon can be regarded as one of the manifestations of the symmetry of the function $\dot{n}(x)$ relative to the middle of the pact (in the case of $N = 2p$).

When every layer is perfectly symmetric, we can show by induction that $R = 0$ for angles $\alpha = \alpha_m$, defined in the assertion 3° of the theorem in Sect. 4. Finally, it is clear that the range of resonances is the same for a pack of layers and for a single layer.

The main result of this work consists of providing the proof that a wave impinging at an angle on an inhomogeneous layer or a pack of identical layers may be transmitted with low energy losses at angles of incidence exceeding the critical angle, i. e. in the region of total internal reflection. Basically analogous results can be found in the paper (*) dealing with the straight and converse problem of scattering of waves on an inhomogeneous layer or a pack of inhomogeneous layers. In particular, the author notes the presence of resonant frequencies in a pack consisting of three layers (a transparent layer between two opaque layers). It should be noted that in this paper points of inflection are assumed to be absent.

The resonance effects discussed above are, generally speaking, very subtle, since their appearance requires matching of the angle of incidence with the properties of the layer (i.e. $\gamma_- = \gamma_+$). Moreover, the resonance band width with respect to frequency (and angle) is exponentially small (see (3.5)).

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